TOPOLOGY - III, EXERCISE SHEET 10

This exercise sheet discusses some applications of the Mayer-Vietoris sequence.

Exercise 1. *Homology of wedge sum.*

(1) Let $(X, x_0), (Y, y_0)$ be pointed spaces such that there exist open neighbourhoods U of x_0 in X and V of y_0 in Y, which deformation retract onto x_0 and y_0 respectively. Show that $\tilde{H}_n(X \vee Y) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$.
(2) Using (1), compute the homology of the wedge sum of g copies of S^1.

Exercise 2. *Homology of Surfaces via Mayer-Vietoris.*
Using Mayer-Vietoris for the open cover from exercise 2, part (2) of sheet 9, compute the homology groups of $(T^2)^{\# n}$ and $(\mathbb{RP}^2)^{\# n}$.

Exercise 3. *Homology of suspension via Mayer-Vietoris*
Recall the definition of the suspension SX of a space X from sheet 6. Show that $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$ for all n using Mayer-Vietoris.

Exercise 4. *A homology vanishing.*
Let X be a topological space and let $U_1, .., U_n$ be an open cover of X such that every U_i is contractible and moreover arbitrary finite intersections of the U_i are contractible. Show that $\tilde{H}_i(X) = 0$ for all $i \geq n - 1$.

Exercise 5. *Homology of Knot Complement.*
Let K be a knot in $\mathbb{R}^3$. That is K is a smooth/piece-wise linear embedding of S^1 in $\mathbb{R}^3$ (you can assume that this is the usual embedding of S^1 in the XY-plane). Compute the homology groups of the so called knot-complement $\mathbb{R}^3 \backslash K$.

Exercise 6. $H_*(X \times S^1)$.
Let X be a topological space, using the long exact sequence for the mapping torus, show that $H_i(X \times S^1) \cong H_i(X) \oplus H_{i-1}(X)$.